B.Sc. Part III Hons. Physics

Paper V

Group (A) — Methods of mathematical Physics

Group (B) — classical - Mechanics

Group (C) — Quantum - mechanics

Group (A)

Methods of mathematical Physics

1) **Volume Integral**: If we consider a closed surface in space enclosing a volume V, then the integral $\iiint \vec{V} \cdot \vec{n} \, dv$ is defined as the volume integral.

2) Field of a Physical quantity :→ A physical quantity can be expressed as a continuous function of the position of point in a region of space. This function is called a point function and the region in which is specifies the physical quantity is known as field; Fields are of two kinds: (a) Scalar & (b) vector, depending upon the nature of the physical quantity.

3) Scalar field :→ A Scalar physical quantity like temperature, density of electric potential can be expressed from ~~pot~~ point to point in a region of space by a continuous scalar point function S which gives the value of the quantity at each point. This region is the Scalar field. The field can be completely mapped out by a series of level surfaces upon each of which S has constant value. As we pass from one surface to the other,

'S' changes by a constant value. Two level surfaces cannot intersect. In other words, 'S' is single valued at each point.

(4) <u>Vector Field</u>: A vector physical quantity such as velocity of magnetic or electric field strength can be expressed from point to point in a region of space by continuous vector function $\vec{V}$. This region is the vector field. At any given point in the field of the function $\vec{V}$ is represented by a vector of definite magnitude and direction, which changes continuously from point to point. The field can be completely mapped out by curved lines, known as "flux lines" or "lines of flow", the tangent at any point of a line giving the direction of $\vec{V}$ at that point. The magnitude of $\vec{V}$ at any point on a flux line is given by the number of flux lines crossing unit area perpendicular to their direction drawn at that point. Two flux lines cannot intersect, thus $\vec{V}$ is single valued at each point.

(5) The Gradient of a scalar field: → Scalar field can be mapped out by a series of level surfaces. The gradient of a scalar field S is a vector field in which the magnitude of the vector at any point is equal to the maximum rate of increase of 'S' at that point and the direction of the vector is along the normal to the level surface at that point. It is given by

$$\text{grad } S = \frac{\partial S}{\partial n}\hat{n}.$$

where $\hat{n}$ is the unit normal vector at that point.

Example: let us suppose that 'S' is the potential in an electric field. The intensity of the field at any point is in the direction of the greatest rate of fall of potential and is equal to this rate.

$$\therefore \vec{E} = \frac{\partial s}{\partial n}\hat{n} = -\text{grad } s$$

$$\text{or, } \text{grad } s = \vec{E} = \frac{\partial s}{\partial n}\hat{n}$$

Grad 's' in terms of cartesian co-ordinates:

Let us suppose that the scalar field s is a function of x, y, z; By the theorem of partial differentiation, we know that

$$ds = \frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy + \frac{\partial s}{\partial z}dz$$

But by defn, $\text{grad } s = \frac{\partial s}{\partial n}\hat{n}$.

which is a vector.

Now taking dot product of grad s with an element of radius vector $d\vec{r}$,

$$\therefore (\text{grad } s)\cdot d\vec{r} = \frac{\partial s}{\partial n}\hat{n}\cdot d\vec{r} = \frac{\partial s}{\partial n}\, n\, dr\cos\theta$$

$$= \frac{\partial s}{\partial n}dr\cos\theta \quad [\because n=1]$$

$$= \frac{\partial s}{\partial n}dn\,(dn = dr\cos\theta)$$

$$= ds = \frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy + \frac{\partial s}{\partial z}dz$$

$$\text{or, } (\text{grad } s)\cdot d\vec{r} = \left(\frac{\partial s}{\partial x}\hat{i} + \frac{\partial s}{\partial y}\hat{j} + \frac{\partial s}{\partial z}\hat{k}\right)\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)S\cdot d\vec{r}$$

But $\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \nabla$, the vector differential operator.

$\therefore \ (\text{grad } s) \cdot d\vec{r} = \nabla s \cdot d\vec{r}$

or, $\text{grad } s = \nabla s = \frac{\partial s}{\partial x}\hat{i} + \frac{\partial s}{\partial y}\hat{j} + \frac{\partial s}{\partial z}\hat{k}$.

(6) Line integral :→ The integral of point function along a curve is called the line integral. Let $r = r(t)$ be the equation of a curve. If ϕ and A are the scalar and vector fields respectively and dr is the vector increment of length, then we may encounter the integrals each of which being known as line integral along the

$$\int_C \phi \, dr \quad \text{---} \quad (1)$$

$$\int_C A \cdot dr \quad \text{---} \quad (2)$$

$$\int_C A \times dr \quad \text{---} \quad (3)$$

~~curve C'~~ curve C that may be open or closed. The results of integration are respectively a vector, a scalar and a vector.

To compute any of the integrals, the method of attack will be to reduce the vector integrals into scalar integrals, with which one is assumed to be familiar.

As ϕ is a scalar function and $dr = i\,dx + j + k\,dz$, integral (1) immediately reduces to

$$\int_C \phi \, dr = \hat{i}\int_C \phi(x,y,z)\,dx + \hat{j}\int_C \phi(x,y,z)\,dy + \hat{k}\int \phi(x,y,z)\,dz$$

The three integrals on R.H.S. of eqn (4) are ordinary scalar integrals and to avoid complications we shall assum that they are Reimann integrals. The integrals with respect to x cannot be evaluated unless y and z are kwon in terms of x and similarly for the integrals with respect to y and z. This simply means that the path of integration C must be specified. Unless the integrand has special properties that lead the integral to depend only on the value of end points, the value will depend upon the particular choice of C.

The line integrals (2) & (3) may be intepreted in the similar fashion and like integral (1) they are dependent, in general, on the choice of the path.

Line integral (2) is most commonly used in vector analysis, it is called the line integral of the tangential component of vector A along the curve C. If A represents the force F acting on the particle, then the line integral. i.e. $\int_C F \cdot dr$ represents the workdone by the force.

An important special case arises in scalar calculus when the function to be integrated is an exact differential where the value of integral is independent of the path. In vector analysis let

$$A = \text{grad}\,\phi = \nabla \phi$$

Then $$\int_P^Q A \cdot dr = \int_P^Q \nabla\phi \cdot dr$$

$$\text{vector} = \int_P^Q \left\{ \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right\}$$

$$= \int_P^Q d\phi = \phi Q - \phi P$$

where P and Q are initial & final points of the curve with coordinates (x_1, y_1, z_1).

If the integration is taken around a closed curve, $Q = P$, so that

$$\int_P^P A \cdot dr = \int_P^P \nabla\phi \cdot dr = \oint \nabla r\, dr = \phi P - \phi P = 0 .$$

where the symbol $\oint$ denotes the integral along the boundary or closed curve.

Thus if $A = \text{grad}\,\phi$, then the line integral $\int_P^Q A \cdot dr$ depends only on initial and final values of ϕ and is independent of the path and also $\oint A \cdot dr = 0$

Conversely, if $\oint A \cdot dr$ is independent of path, then the vector field $A(x, y, z)$ is said to be conservative or lameller or non-curl vector field.

From above discussion it follows that

A vector field A is said to be conservative if and only if there exists a scalar point function ϕ such that $A = \text{grad}\,\phi$.

conversely if a vector field A is derivable from a scalar point function ϕ according to relation $A = \text{grad}\,\phi$, then vector field A is a conservative field.

(7) <u>Surface Integral</u> ⇒ Let S be any surface, divided into infinitesimal elements each of which may be considered as a vector ds. Then if ϕ and A are scalar and vector fields respectively, the surface integrals may be encountered

as $\iint_S \phi \, ds$ ---- (5)

$\iint_S A \cdot ds$ ---- (6)

$\iint_S A \times d$ ---- (7)

The results of intgration are respectively a vector, a scalar and a vector. Often the area element ds may be written as $ds - \hat{n} \, ds$ where $\hat{n}$ is a unit vector to indicate the positive direction. There are two conventions for choosing the positive direction; first, if the surface is closed surface, we agree to take the outword normal as positive. Secondly, if the surface is an open surface the positive normal depends on the direction in which the perimeter of the open surface is traversed. If the right hand fingers are placed in the direction of travel around the perimeter, the positive normal is indicated by the thumb of the right hand.

The surface integral $\iint_S A \cdot ds$ is called the flow or flux of the vector function A through the given surface S. For example, if $A = \rho v$ where ρ is the density and v the velocity of fluid, then the surface intgral $\iint_S \rho v \cdot ds$ denotes the amount of fluid flowing the given surface in unit time.

—x—

By. Dr. Sanjay Kumar
Department of Physics
S.S. College Jehanabad.